

PICARD-VESSIOT EXTENSIONS FOR UNIPOTENT ALGEBRAIC GROUPS

V. RAVI SRINIVASAN

ABSTRACT. Let F be a differential field of characteristic zero. In this article, we construct Picard-Vessiot extensions of F whose differential Galois group is isomorphic to the full unipotent subgroup of the upper triangular group defined over the field of constants of F . We will also give a procedure to compute linear differential operators for our Picard-Vessiot extensions. We do not require the condition that the field of constants be algebraically closed.

1. INTRODUCTION

Throughout this article, we fix a ground differential field F of characteristic zero. All the differential fields considered henceforth are either differential subfields of F or a differential field extension of F . We deal with differential fields equipped with only one derivation and we reserve the notation $'$ to denote that derivation map. The author assumes that the reader is familiar with the notion of a differential field and Picard-Vessiot theory. For precise definitions see [4] and [5].

Let $U(n, C)$ be the subgroup of $GL(n, C)$ of all upper triangular matrices with 1's on the diagonal. In this article, we describe a procedure to compute linear homogeneous differential equations over F for the group $U(n, C)$. Here is the statement of our main result:

THEOREM 1.1. *Let F be a differential field and C be its field of constants. Suppose that F contains distinct elements f_1, f_2, \dots, f_n satisfying the following condition:*

- (C) *if there are elements $c_1, c_2, \dots, c_n \in C$ and an element $f \in F$ such that $\sum_{i=1}^n c_i f_i = f'$ then $c_i = 0$ for all i .*

Let $\mathfrak{S} := \{\zeta_{i,j} | 1 \leq i \leq n, 1 \leq j \leq n+1-i\}$ be a set of $n(n+1)/2$ -variables and let $E := F(\mathfrak{S})$ be the field of rational functions in these variables over

the field F . Let

$$(1.1) \quad g := \begin{pmatrix} 1 & \zeta_{1,1} & \zeta_{2,1} & \cdots & \zeta_{n,1} \\ 0 & 1 & \zeta_{1,2} & \cdots & \zeta_{n-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \zeta_{1,n} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad A := \begin{pmatrix} 0 & f_1 & 0 & \cdots & 0 \\ 0 & 0 & f_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & f_n \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Extend the derivation of F to E by setting

$$(1.2) \quad g' = Ag.$$

Then E is a Picard-Vessiot extension of F for the differential operator

$$(1.3) \quad L(Y) := \frac{w(Y, 1, \zeta_{1,1}, \zeta_{2,1}, \dots, \zeta_{n,1})}{w(1, \zeta_{1,1}, \zeta_{2,1}, \dots, \zeta_{n,1})},$$

$V := \text{span}_C\{\zeta_{1,1}, \zeta_{2,1}, \dots, \zeta_{n,1}\}$ is the full set of solutions of the equation $L(Y) = 0$, and the differential Galois group $G := G(E|F)$ is naturally isomorphic to the group $U(n+1, C)$.

When $F = \mathbb{C}(z)$ with the usual derivation d/dz , a maple program to compute equation 1.3 can be found in my website [10].

Under the conditions that the field of constants C of F be algebraically closed and that F has a non zero but finite transcendence degree over C , Biyalinicki-Birula [2] has shown that every connected nilpotent algebraic group defined over C can be realized as a differential Galois group. And, in [6] the authors construct Picard-Vessiot extensions of F for connected linear algebraic groups defined over C . Thus, in particular, the existence of a Picard-Vessiot extension for the group $U(n+1, C)$ is known. Our approach differs and gives a different perspective from the above mentioned articles.

Acknowledgements. The author would like to thank the participants of the Kolchin Seminar, especially Phyllis Cassidy and William Sit, for their suggestions and criticisms on this work. Special thanks to Jonathan Sparling and to William Keigher for their encouragement and support throughout this project.

1.1. Preliminaries. Here we will state some results that will be used often in this article. The following theorems aid us in constructing no new constants extension by adjoining antiderivatives, see [4] or [8].

THEOREM 1.2. *Let F be a differential field and let $E = F(\zeta)$ be a differential field extension of F such that $\zeta' \in F$. Suppose that E has a new constant. Then there is a $y \in F$ such that $y' = \zeta'$.*

The next theorem characterizes the algebraic dependence of antiderivatives and it is a special case of the Kolchin-Ostrowski theorem (see [1], [7]).

THEOREM 1.3. *Let $E \supset F$ be a no new constants extension and for $i = 1, 2, \dots, n$, let $\zeta_i \in E$ be antiderivatives of F . Then either ζ_i 's are algebraically independent over F or there is a tuple $(c_1, \dots, c_n) \in C^n - \{0\}$ such that $\sum_{i=1}^n c_i \zeta_i \in F$.*

A direct consequence of the above theorem is the following proposition, see [9].

Proposition 1.4. *Let $E = F(\zeta_1, \zeta_2, \dots, \zeta_t)$ be an antiderivative extension of F . An element $\zeta \in E$ is an antiderivative of F if and only if there are a tuple $(c_1, \dots, c_t) \in C^t$ and an element $f \in F$ such that $\zeta = \sum_{i=1}^t c_i \zeta_i + f$.*

THEOREM 1.5. *Let F be a differential field and suppose that there are elements $f_1, f_2, \dots, f_n \in F$ satisfying the condition **C**. Let $E = F(\zeta_1, \zeta_2, \dots, \zeta_n)$ be the field of rational functions over F in variables $\zeta_1, \zeta_2, \dots, \zeta_n$. Extend the derivation of F to E by defining $\zeta'_i = f_i$. Then E is a no new constants extension of F .*

Proof. Let $F_0 := F$ and $F_i := F_{i-1}(\zeta_i)$. Suppose that the theorem is false. Then pick the smallest k such that F_k has a new constant. Note that $F_k = F_{k-1}(\zeta_k)$ and that there is an $a \in F_k - F_{k-1}$ such that $a' = 0$. Therefore, by theorem 1.2, there is a $y \in F_{k-1}$ such that $y' = f_k \in F$. Now we apply proposition 1.4 and obtain that $y = \sum_{i=1}^{k-1} \alpha_i \zeta_i + f$. Taking derivatives we obtain $\sum_{i=1}^k c_i f_i = f'$, where $c_1 := 1$ and $c_i = -\alpha_i$ for all $i \geq 1$. This contradicts the choice of f_i 's. \square

Remark 1. *Let $F = \mathbb{C}(z)$ be the ordinary differential field of rational functions in one complex variable z with the derivation d/dz . For any rational function $f \in F$, $\frac{df}{dz}$ has no simple pole and thus for any distinct complex numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ and for any choice of constants c_1, c_2, \dots, c_n , not all zero, there is no rational function $f \in F$ such that $\frac{df}{dz} = \sum_{i=1}^n c_i / (z + \alpha_i)$. Therefore, for the differential field $\mathbb{C}(z)$, elements $1/(z + \alpha_1), 1/(z + \alpha_2), \dots, 1/(z + \alpha_n)$ satisfy condition **C**.*

2. PICARD-VESSIOT EXTENSIONS FOR $U(3, C)$

In this section, we will provide a construction of a Picard-Vessiot extension for the group $U(3, C)$.

Proposition 2.1. *Let K be a differential field with a field of constants C and let $K(\zeta)$ be a no new constants extension of K such that ζ is transcendental over K and $\zeta' \in K$. Let $S = \sum_{i=0}^s q_i \zeta^i \in K[\zeta]$, $q_i \in K$ and $q_s \neq 0$.*

1. *If there is a $T \in K(\zeta)$ such that $T' = S$ then $T \in K[\zeta]$ and its degree, $\deg T$, equals s or $s + 1$.*
2. *And if $c\zeta' + f' \neq q_s$ for any $c \in C$ and for any $f \in K$ then there is no $T \in K(\zeta)$ such that $T' = S$.*

Proof. Let there be an element $T \in K(\zeta)$ such that $T' = S$. Then there are relatively prime polynomials $P, Q \in K[\zeta]$, where Q is monic, such that $T = P/Q$. Taking derivatives, we obtain

$$(2.1) \quad Q^2 S = P'Q - Q'P.$$

From the above equation, it is immediate that Q divides Q' . On the other hand, since Q is monic and $\zeta' \in K$ we know that $\deg Q' < \deg Q$. Therefore $Q = 1$ and thus $P = T \in K[\zeta]$. Let $T = \sum_{i=0}^t r_i \zeta^i$, $r_t \neq 0$. From equation 2.1, we have

$$(2.2) \quad r'_t \zeta^t + (tr_t \zeta' + r'_{t-1}) \zeta^{t-1} + \cdots + r_1 \zeta' + r'_0 = \sum_{i=0}^s q_i \zeta^i.$$

Since $q_s \neq 0$, t cannot be smaller than s . If $t \geq s + 2$ then $r'_t = 0$ and $tr_t \zeta' + r'_{t-1} = 0$. Then $tr_t \zeta + r_{t-1} \in C \subset K$, contradicting the fact that ζ is transcendental over K . Thus $t = s$ or $t = s + 1$.

Furthermore, if $t = s$ then $r'_t = q_s$, where $r_t \in K$. And if $t = s + 1$ then $r'_t = 0$ and $tr_t \zeta' + r'_{t-1} = q_s$. Thus, we have shown that if $c\zeta' + f' \neq q_s$ for any $c \in C$ and for any $f \in K$ then there is no $T \in K(\zeta)$ such that $T' = S$. \square

THEOREM 2.2. *Let F be a differential field with a field of constants C . Suppose that $f_1, f_2 \in F$ be elements satisfying the condition **C**. Let $E := F(\zeta_1, \zeta_2)(\eta)$ be the field of rational functions of F in three variable ζ_1, ζ_2 and η . Choose any $r \in F$ and extend the derivation of F to E by setting*

$$(2.3) \quad \begin{pmatrix} 1 & \zeta_1 & \eta \\ 0 & 1 & \zeta_2 \\ 0 & 0 & 1 \end{pmatrix}' = \begin{pmatrix} 0 & f_1 & r \\ 0 & 0 & f_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \zeta_1 & \eta \\ 0 & 1 & \zeta_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then E is a no new constants extension of F . In particular, there is no $y \in F(\zeta_1, \zeta_2)$ such that $y' = \eta' = f_1 \zeta_2 + r$ for any $r \in F$.

Proof. Suppose that the theorem is not true for some $r \in F$. Since $\zeta'_1 = f_1$ and $\zeta'_2 = f_2$, from theorem 1.5, we know that $F(\zeta_1, \zeta_2)$ is a no new constants extension of F . Thus there is a new constant in the set $E - F(\zeta_1, \zeta_2)$. Note that $\eta' = f_1 \zeta_2 + r \in F(\zeta_2) \subset F(\zeta_1, \zeta_2)$ and that $E = F(\zeta_1, \zeta_2)(\eta)$. Therefore, by theorem 1.2, there is a $\mu \in F(\zeta_1, \zeta_2)$ such that $\mu' = f_1 \zeta_2 + r$.

Now, we apply proposition 2.1 (with $K = F(\zeta_1)$) and obtain that $\mu \in F(\zeta_1)[\zeta_2]$ and that $\deg \mu = 1$ or 2 .

Case $\deg \mu = 2$: Let $\mu = r_2 \zeta_2^2 + r_1 \zeta_2 + r_0 \in F(\zeta_1)[\zeta_2]$ with $r_2 \neq 0$. Then taking derivatives, we obtain $r'_2 = 0$ and $2r_2 f_2 + r'_1 = f_1$. This contradicts the condition **C**.

Case $\deg \mu = 1$: Let $\mu = r_1\zeta_2 + r_0$. Then we have $r'_1 = f_1$ and $r_1f_2 + r'_0 = r$. Thus there is a constant $c \in C$ such that $r_1 = \zeta_1 + c$ and therefore $(\zeta_1 + c)f_2 + r'_0 = r$.

Now we have

$$(2.4) \quad f_2\zeta_1 + s = -r'_0,$$

where $r_0 \in F(\zeta_1)$ and $s = cf_2 - r \in F$. Then by proposition 2.1 (2), there are a constant $c \in C$ and an element $f \in F$ such that $cf_1 + f' = f_2$. This again contradicts the condition **C**. \square

Corollary 2.2.1. *Let F and E be as in theorem 2.2 and let $\zeta \in E$ be an antiderivative of F . Then there are constants $c_1, c_2 \in C$ and an element $s \in F$ such that $\zeta = c_1\zeta_1 + c_2\zeta_2 + s$.*

Proof. By proposition 1.4, there is a constant $c \in C$ such that $\zeta + c\eta \in F(\zeta_1, \zeta_2)$. We claim that $c = 0$. Suppose not. Then, $c^{-1}\zeta + \eta \in F(\zeta_1, \zeta_2)$ and let $s := c^{-1}\zeta' + r \in F$. Let $E^* := F(\zeta_1, \zeta_2)(\mu)$ be the field of rational functions in one variable μ and extend the derivation of $F(\zeta_1, \zeta_2)$ to the field E^* by setting $\mu' = f_1\zeta_2 + s$. But since $c^{-1}\zeta + \eta \in F(\zeta_1, \zeta_2)$ and $(c^{-1}\zeta + \eta)' = f_1\zeta_2 + s$, we obtain that E^* has a new constant, namely, $\mu - (c^{-1}\zeta + \eta)$. This contradicts theorem 2.2.

Thus $\zeta \in F(\zeta_1, \zeta_2)$. Now we again apply proposition 1.4 to prove the corollary. \square

In theorem 2.5, we will prove that the differential field E , as described in theorem 2.2, is a Picard-Vessiot extension with a differential Galois group isomorphic to $U(3, C)$ as groups. In order to do so, we will require the following two propositions to prove theorem 2.5.

Proposition 2.3. *Let F be a field and $E = F(\zeta_{i,j} | 1 \leq i \leq n, 1 \leq j \leq n+1-i)$ be the field of rational functions over $n(n+1)/2$ variables. Let g*

$$\text{be as in theorem 1.1. For } M := \begin{pmatrix} 1 & c_{1,1} & c_{2,1} & \cdots & c_{n,1} \\ 0 & 1 & c_{1,2} & \cdots & c_{n-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c_{1,n} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in U(n+1, F)$$

let $G := \{\sigma_M : E \rightarrow E | M \in U(n+1, F)\}$ be a collection of automorphisms on E defined by $\sigma_M(g) = gM$. That is,

$$(2.5) \quad \sigma(\zeta_{i,j}) = \begin{cases} \zeta_{i,j} + \left(\sum_{t=1}^{i-1} c_{t,i+j-t} \zeta_{i-t,j} \right) + c_{i,j}, & \text{if } i \geq 2; \\ \zeta_{1,j} + c_{1,j}, & \text{if } i = 1. \end{cases}$$

Then E^G , the field fixed by G , equals F .

Proof. Let $S_{p,q} := \{\zeta_{i,j} | 1 \leq i \leq p-1\} \cup \{\zeta_{p,j} | 1 \leq j \leq q-1\}$ and let $K_{p,q} := F(S_{p,q})$. Let $u \in E - F$ and choose the largest integer p and a largest integer q such that $u \in K_{p,q}(\zeta_{p,q})$. Then, there are relatively prime polynomials $P, Q \in K_{p,q}[\zeta_{p,q}]$ such that $u = P/Q$. Consider a matrix $M \in U(n+1, F)$ such that $c_{i,j} = 0$ for all $i \neq p$ and $j \neq q$, and $c_{p,q} \neq 0$. Then we see that $\sigma_M(\zeta_{p,q}) = \zeta_{p,q} + c_{p,q}$ and $\sigma_M(\zeta_{i,j}) = \zeta_{i,j}$ for all $1 \leq i \leq p$ and $j \leq q-1$. Thus, in particular, the field $K_{p,q}$ is fixed by the automorphism σ_M .

Suppose that $\sigma_M(u) = u$. Then $\sigma_M(P)Q = \sigma_M(Q)P$ and since P, Q are relatively prime, there is an element $r_{\sigma_M} \in K$ such that $\sigma_M(P(\zeta_{p,q})) = r_{\sigma_M}P(\zeta_{p,q})$ and $\sigma_M(Q(\zeta_{p,q})) = r_{\sigma_M}Q(\zeta_{p,q})$. That is $P(\zeta_{p,q} + c_{p,q}) = r_{\sigma_M}P(\zeta_{p,q})$ and $Q(\zeta_{p,q} + c_{p,q}) = r_{\sigma_M}Q(\zeta_{p,q})$. But these equations hold only when $c_{p,q} = 0$, a contradiction. \square

Proposition 2.4. *Let E be a differential field extension of F . Let $y_1, y_2, \dots, y_n \in E$ and let $V := \text{Span}_{C_E}\{y_1, y_2, \dots, y_n\}$. Suppose that the wronskian $w(y_1, y_2, \dots, y_n) \neq 0$ and that the group $G := G(E|F)$ of all differential automorphisms of E fixing F stabilizes the vector space V . Then the differential operator $L(Y) := \frac{w(Y, y_1, y_2, \dots, y_n)}{w(y_1, y_2, \dots, y_n)}$ has coefficients in the differential field E^G . Moreover, V is the full set of solutions of the differential equation $L(Y) = 0$.*

Proof. Write $L(Y) = Y^{(n)} + a_{n-1}Y^{(n-1)} + \dots + a_1Y^{(1)} + a_0Y$. We will show that $a_i \in E^G$ for each i . Clearly, V is the full set of solutions of $L(Y) = 0$. Let $L_\sigma := Y^{(n)} + \sigma(a_{n-1})Y^{(n-1)} + \dots + \sigma(a_1)Y^{(1)} + \sigma(a_0)Y$ for $\sigma \in G$. Since G stabilizes V and that V is finite dimensional, we obtain that G consists of automorphisms of the vector space V . It follows that $\ker(L) = \ker(L_\sigma) = V$, in particular $\ker(L - L_\sigma) \supset V$, and thus the dimension of $\ker(L - L_\sigma) \geq n$. Since $(L - L_\sigma)(Y) = (\sigma(a_{n-1}) - a_{n-1})Y^{(n-1)} + \dots + (\sigma(a_1) - a_1)Y^{(1)} + (\sigma(a_0) - a_0)Y$ is of order $\leq n-1$, we should have $L - L_\sigma = 0$. That is, $a_i \in E^G$ for all $i, 0 \leq i \leq n-1$. \square

THEOREM 2.5. *Let E and F be differential fields as defined in theorem 2.2. Then E is a Picard-Vessiot extension of F for the differential operator*

$$(2.6) \quad L(Y) := \frac{w(Y, 1, \zeta_1, \eta)}{w(1, \zeta_1, \eta)},$$

and $L^{-1}(0) = V$. And, the differential Galois group $G := G(E|F)$ is naturally isomorphic to the group $U(3, C)$.

Proof. From theorem 2.2, we know that E is a no new constants extension of F . Let $R := F[\zeta_1, \zeta_2, \eta]$ and let $\sigma \in G := G(E|F)$. Since $\sigma(\zeta_i)' = \sigma(\zeta_i')$, we have $\sigma(\zeta_1) = \zeta_1 + \alpha_\sigma$ and $\sigma(\zeta_2) = \zeta_2 + \beta_\sigma$, where $\alpha_\sigma, \beta_\sigma \in C$. And since $\sigma(\eta)' = \sigma(\eta') = \sigma(\zeta_1')\sigma(\zeta_2) + \sigma(f) = \zeta_1'(\zeta_2 + \beta_\sigma) + f =$

$(\eta + \beta_\sigma \zeta_1)'$, there is a $\gamma_\sigma \in C$ such that $\sigma(\eta) = \eta + \beta_\sigma \zeta_1 + \gamma_\sigma$. Thus if $V := \text{Span}_C\{1, \zeta_1, \eta\}$ then $GV \subseteq V$. Let $g := \begin{pmatrix} 1 & \zeta_1 & \eta \\ 0 & 1 & \zeta_2 \\ 0 & 0 & 1 \end{pmatrix}$ and $M_\sigma := \begin{pmatrix} 1 & \alpha_\sigma & \gamma_\sigma \\ 0 & 1 & \beta_\sigma \\ 0 & 0 & 1 \end{pmatrix}$. Then we have $\sigma(g) = gM_\sigma$ and with respect to the basis $\{1, \zeta_1, \eta\}$, we have a group representation $\Gamma : G \rightarrow GL(3, C)$ defined by $\Gamma(\sigma) = M_\sigma$. Note that $F\langle V \rangle$, the differential field generated by F and V , equals E . Therefore the action of σ on the elements ζ_1 and η completely determines the differential automorphism σ on E . Thus Γ is a faithful representation of groups. Given a matrix $M := \begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \in U(3, C)$, we define a F -algebra automorphism $\sigma_M : R \rightarrow R$ such that $\sigma_M(g) = gM$. Since $\sigma_M(\zeta_i)' = \sigma_M(\zeta_i')$ for $i = 1, 2$ and $\sigma_M(\eta)' = \sigma_M(\eta')$, we see that σ_M is an F -algebra differential automorphism of R . Now one extends σ_M to E , the field of fractions of R , to obtain a differential field automorphism of E . Hence $\Gamma(G)$ is isomorphic to $U(3, C)$.

We note that the differential operator $L(Y) := \frac{w(Y, 1, \zeta_1, \eta)}{w(1, \zeta_1, \eta)}$ has $\text{span}_C\{1, \zeta_1, \eta\}$ as the full set of solutions. From proposition 2.3 and 2.4, we know that the coefficients of the differential operator $L(Y)$ lie in the field F . Thus E is a Picard-Vessiot extension of F with Galois group isomorphic to $U(3, C)$. \square

Example 2.1. Let $F = \mathbb{C}(z)$ with the derivation d/dz and let α_1, α_2 be two distinct complex numbers. Let $f_i = 1/(z + \alpha_i)$ for $i = 1, 2$, $r = 0$ and let $E = F(\zeta_1, \zeta_2, \eta)$ be the field of rational functions in three variables ζ_1, ζ_2, η . Extend the derivation of F to E using equation 2.3. Now we apply theorem 2.5 and obtain that E is a Picard-Vessiot extension of F for the differential operator

$$(2.7) \quad L := \frac{d^3}{dz^3} + \frac{3z + \alpha_1 + 2\alpha_2}{(z + \alpha_1)(z + \alpha_2)} \frac{d^2}{dz^2} + \frac{1}{(z + \alpha_1)(z + \alpha_2)} \frac{d}{dz},$$

whose solution space $L^{-1}(0) = \text{Span}_\mathbb{C}\{1, \zeta_1, \eta\}$, see [10]. The differential Galois group of E is isomorphic to the Heisenberg group $U(3, C)$. One can think of ζ_1, ζ_2 and η as $\log(z + \alpha_1)$, $\log(z + \alpha_2)$ and $\int_{a_0}^z \frac{\log(t + \alpha_2)}{t + \alpha_1} dt$, $a_0 \neq \alpha_1$ respectively. The integral $\int_{a_0}^z \frac{\log(t + \alpha_2)}{t + \alpha_1} dt$ is a ‘shifted’ dilogarithm.

3. PICARD-VESSIOT EXTENSIONS FOR $U(n, C)$

THEOREM 3.1. Let F and E differential fields as in theorem 1.1. Then

- a. The differential field $F\langle\zeta_{i,j}\rangle$ equals the field $F(\zeta_{1,i+j-1}, \zeta_{2,i+j-2}, \dots, \zeta_{i-1,j+1}, \zeta_{i,j})$. In particular, $E = F\langle\zeta_{1,1}, \zeta_{2,1}, \dots, \zeta_{n,1}\rangle$.
- b. The differential field $N := F\langle\zeta_{n,1}\rangle(\zeta_{1,1}, \dots, \zeta_{1,n})$ is a no new constants extension of F .
- c. E is a no new constants extension of F .

Proof. From equation 1.2, we obtain

$$(3.1) \quad \zeta'_{1j} = f_j$$

$$(3.2) \quad \zeta'_{ij} = f_j \zeta_{i-1,j+1}, \quad 2 \leq i \leq n \text{ and } 1 \leq j \leq n+1-i.$$

(a): Therefore $f_j \zeta_{i-1,j+1} = \zeta'_{ij} \in F\langle\zeta_{i,j}\rangle$. And since $f_j \in F$, we obtain $\zeta_{i-1,j+1} \in F\langle\zeta_{i,j}\rangle$. Repeating this argument, one proves that $F\langle\zeta_{i,j}\rangle \supseteq F(\zeta_{1,i+j-1}, \zeta_{2,i+j-2}, \dots, \zeta_{i-1,j+1}, \zeta_{i,j})$. On the other hand, equations 3.1 and 3.2 also tell us that $F(\zeta_{1,i+j-1}, \zeta_{2,i+j-2}, \dots, \zeta_{i-1,j+1}, \zeta_{i,j})$ is a differential field. Since $F\langle\zeta_{i,j}\rangle$ is the smallest differential field containing F and $\zeta_{i,j}$, we obtain $F\langle\zeta_{i,j}\rangle = F(\zeta_{1,i+j-1}, \zeta_{2,i+j-2}, \dots, \zeta_{i-1,j+1}, \zeta_{i,j})$. It is easy to check that $E = F\langle\zeta_{1,1}, \zeta_{2,1}, \dots, \zeta_{n,1}\rangle$.

(b): Let $N_k := F(\zeta_{1,n-k}, \dots, \zeta_{1,n})(\zeta_{2,n-1}, \dots, \zeta_{k+1,n-k})$ and observe from statement (a) that $N_k = F(\zeta_{1,n-k}, \dots, \zeta_{1,n})\langle\zeta_{k+1,n-k}\rangle$. We see from theorem 2.2 that N_1 is a no new constants extension of F . Assume that N_k is a no new constants extension of F for some $k \geq 1$. Let $K = F(\zeta_{1,n-k}, \dots, \zeta_{1,n})(\zeta_{2,n-1}, \dots, \zeta_{k,n-(k-1)})$ and note that $N_{k+1} = K(\zeta_{1,n-(k+1)}, \zeta_{k+1,n-k})(\zeta_{k+2,n-(k+1)})$. Applying theorem 2.2 we obtain N_{k+1} is a no new constants extension of K . Since $K \subset N_k$, we see that K is a no new constants extension of F . Thus N_k is a no new constants extension of F . Choose $k = n$ to prove statement (b).

(c): The case $n = 2$ follows from theorem 2.2. Assume that (c) is true for some $n \geq 3$ and let $F^* := F\langle\zeta_{n,1}\rangle$,

$$g^* := \begin{pmatrix} 1 & \zeta_{1,1} & \zeta_{2,1} & \cdots & \zeta_{n-1,1} \\ 0 & 1 & \zeta_{1,2} & \cdots & \zeta_{n-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \zeta_{1,n} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \text{ and } A^* := \begin{pmatrix} 0 & f_1 & 0 & \cdots & 0 \\ 0 & 0 & f_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & f_{n-1} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Then from (a) it follows that $E = F^*(g^*)$. Since $(g^*)' = A^*g^*$, applying induction, we obtain that E is a no new constants extension of F^* . From (b) we know that N is a no new constants extension of F . In particular, $F^* \subset N$ is a no new constants extension of F as well. Thus we have shown that E is a no new constants extension of F . \square

Corollary 3.1.1. *Let $\zeta \in N$ be an antiderivative of F . Then, there are constants $c_1, \dots, c_n \in C_F$ and an element $f \in F$ such that $\zeta = \sum_{i=1}^n c_i \zeta_i + f$. Moreover, if $\zeta \in F\langle \zeta_{1,n} \rangle$ then $c_i = 0$ for all $i \leq n-1$.*

Proof. From corollary 2.2.1 it is enough to consider the case when $n \geq 3$. Assume the corollary for the field $N^* := F(\zeta_{1,2}, \dots, \zeta_{1,n})\langle \zeta_{n-1,2} \rangle$. Let $K := F(\zeta_{1,2}, \dots, \zeta_{1,n})\langle \zeta_{n-2,3} \rangle$ and note that $K \subset N^*$. Since $\zeta \in N = K(\zeta_{1,1}, \zeta_{n-1,2})\langle \zeta_{n,1} \rangle$ and $\zeta' \in F \subset K$, by corollary 2.2.1, there are constants $c_1, d_1 \in C$ such that $\zeta - c_1 \zeta_{1,1} - d_1 \zeta_{n-1,2} \in K \subset N^*$. Since $\zeta_{n-1,2} \in N^*$, we have $\zeta - c_1 \zeta_{1,1} \in N^*$. Note that $\zeta - c_1 \zeta_{1,1} \in N^*$ is an antiderivative of F and therefore, applying induction to the differential field N^* , we obtain constants c_2, \dots, c_n such that $\zeta - c_1 \zeta_{1,1} = \sum_{i=2}^n c_i \zeta_{1,i} + f$ for some $f \in F$ as desired. From (a), we know that $F\langle \zeta_{1,n} \rangle = F(\zeta_{1,n}, \zeta_{2,n-1}, \dots, \zeta_{n,1})$ and thus $\zeta_{1,1}, \dots, \zeta_{1,n-1}$ remains algebraically independent over $F\langle \zeta_{1,n} \rangle$. Hence $\zeta = \sum_{i=1}^n c_i \zeta_{1,i} + f \in F\langle \zeta_{1,n} \rangle$ implies $c_i = 0$ for all $i \leq n-1$. \square

Corollary 3.1.2. *Let $\zeta \in E$ be an antiderivative of F . Then there are constants $c_1, \dots, c_n \in C$ and an $f \in F$ such that $\zeta = \sum_{i=1}^n c_i \zeta_i + f$. Moreover, if $\zeta \in F\langle \zeta_{s,t} \rangle$ then $c_i = 0$ for all $i \neq s+t-1$ and thus $\zeta = c_{s+t-1} \zeta_{1,s+t-1} + f$.*

Proof. Since $\zeta \in E = F^*(g^*)$, applying induction with $F\langle \zeta_{n,1} \rangle$ as our base field, we obtain constants $c_1, \dots, c_{n-1} \in C$ and an $\tilde{f} \in F\langle \zeta_{n,1} \rangle$ such that

$$(3.3) \quad \zeta = \sum_{i=1}^{n-1} c_i \zeta_{1,i} + \tilde{f}.$$

Then $\tilde{f} = \zeta - \sum_{i=1}^{n-1} c_i \zeta_{1,i} \in F\langle \zeta_{n,1} \rangle$ is an antiderivative of F . Now we apply corollary 3.1.1 to the differential field $F\langle \zeta_{n,1} \rangle$ and obtain a constant $c_n \in C$ and an element $f \in F$ such that $\tilde{f} = c_n \zeta_{1,n} + f$. Substituting back for \tilde{f} in equation 3.3, we obtain $\zeta = \sum_{i=1}^n c_i \zeta_{1,i} + f$.

Note that $\zeta_{1,1}, \dots, \zeta_{1,s+t-2}, \zeta_{1,s+t}, \dots, \zeta_{1,n}$ remains algebraically independent over $F\langle \zeta_{s,t} \rangle$, see (a). Thus $\zeta = \sum_{i=1}^n c_i \zeta_{1,i} + f \in F\langle \zeta_{s,t} \rangle$ implies that $c_i = 0$ for all $i \neq s+t-1$. \square

Proof of theorem 1.1. We know that E is a no new constants extension of F . For $\sigma \in G$ and for each $1 \leq j \leq n$, we have $\sigma(\zeta_{1,j}) = \zeta_{1,j} + c_{1,j}^\sigma$ for some constants $c_{1,j}^\sigma \in C$. For $1 \leq j \leq n-1$, we have $\zeta'_{2,j} = f_j \zeta_{1,j+1}$ and therefore $\sigma(\zeta'_{2,j})' = \sigma(\zeta'_{2,j}) = f_j \sigma(\zeta_{1,j+1}) = f_j \zeta_{1,j+1} + f_j c_{1,j+1}^\sigma = (\zeta_{2,j} + c_{1,j+1}^\sigma \zeta_{1,j})'$. Thus, there are constants $c_{2,j}^\sigma \in C$ such that $\sigma(\zeta_{2,j}) = \zeta_{2,j} + c_{1,j+1}^\sigma \zeta_{1,j} + c_{2,j}^\sigma$, for all $1 \leq j \leq n-2$. Assume that for some integer $s, 2 \leq s \leq n$, there are constants $c_{i,j}^\sigma \in C$ such that

$$\sigma(\zeta_{s,j}) = \zeta_{s,j} + \left(\sum_{t=1}^{s-1} c_{t,s+j-t}^\sigma \zeta_{s-t,j} \right) + c_{s,j}^\sigma,$$

for all $1 \leq j \leq n+1-s$. Note that

$$\begin{aligned} \sigma(\zeta_{s+1,j})' &= f_j \sigma(\zeta_{s,j+1}) \\ &= f_j \left(\zeta_{s,j+1} + \left(\sum_{t=1}^{s-1} c_{t,s+1+j-t}^\sigma \zeta_{s-t,j+1} \right) + c_{s,j+1}^\sigma \right) \\ &= \left(\zeta_{s+1,j} + \left(\sum_{t=1}^{s-1} c_{t,s+1+j-t}^\sigma \zeta_{s+1-t,j} \right) + c_{s,j+1}^\sigma \zeta_{1,j} \right)' \end{aligned}$$

and thus there is a $c_{s+1,j}^\sigma \in C$ such that

$$\begin{aligned} \sigma(\zeta_{s+1,j}) &= \zeta_{s+1,j} + \left(\sum_{t=1}^{s-1} c_{t,s+1+j-t}^\sigma \zeta_{s+1-t,j} \right) + c_{s,j+1}^\sigma \zeta_{1,j} + c_{s+1,j}^\sigma, \\ &= \zeta_{s+1,j} + \left(\sum_{t=1}^s c_{t,s+1+j-t}^\sigma \zeta_{s+1-t,j} \right) + c_{s+1,j}^\sigma. \end{aligned}$$

Then, by induction, for any fixed $i, 1 \leq i \leq n$ and for all $j, 1 \leq j \leq n+1-i$ there are constants $c_{i,j}^\sigma \in C$ such that

$$(3.4) \quad \sigma(\zeta_{i,j}) = \begin{cases} \zeta_{i,j} + \left(\sum_{t=1}^{i-1} c_{t,i+j-t}^\sigma \zeta_{i-t,j} \right) + c_{i,j}^\sigma, & \text{if } i \geq 2; \\ \zeta_{1,j} + c_{1,j}^\sigma, & \text{if } i = 1. \end{cases}$$

Thus, if $g := \begin{pmatrix} 1 & \zeta_{1,1} & \zeta_{2,1} & \cdots & \zeta_{n,1} \\ 0 & 1 & \zeta_{1,2} & \cdots & \zeta_{n-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \zeta_{1,n} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$ and $\sigma \in G$ then, from equation

3.4, we see that there is an element

$$M_\sigma := \begin{pmatrix} 1 & c_{1,1}^\sigma & c_{2,1}^\sigma & \cdots & c_{n,1}^\sigma \\ 0 & 1 & c_{1,2}^\sigma & \cdots & c_{n-1,2}^\sigma \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c_{1,n}^\sigma \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in U(n+1, C)$$

such that $\sigma(g) = gM_\sigma$.

Let $V := \text{Span}_C\{1, \zeta_{1,1}, \zeta_{2,1}, \dots, \zeta_{n,1}\}$. From equation 3.4, it is clear that $GV \subset V$. And, with respect to the basis $1, \zeta_{1,1}, \zeta_{2,1}, \dots, \zeta_{n,1}$, we have a representation $\Gamma : G \rightarrow GL(n+1, C)$ defined by $\Gamma(\sigma) = M_\sigma$. And indeed, $\Gamma(G) \subseteq U(n+1, C)$. Note that if $\sigma, \rho \in G$ agrees on V , then they agree on

$F\langle V \rangle = E$ and thus Γ is injective. For any matrix

$$(3.5) \quad M := \begin{pmatrix} 1 & c_{1,1} & c_{2,1} & \cdots & c_{n,1} \\ 0 & 1 & c_{1,2} & \cdots & c_{n-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c_{1,n} \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in U(n+1, C)$$

let σ_M be the F -algebra automorphism on $F[g]$ defined by $\sigma_M(g) = gM$. Note that $\sigma_M \sigma_{M^{-1}} = \sigma_{M^{-1}} \sigma_M = I$, the identity matrix. Thus σ_M is a ring automorphism of $F[g]$. To show that σ_M is a differential automorphism, we only need to check that $\sigma_M(\zeta_{i,j})' = \sigma_M(\zeta'_{i,j})$. Since $\sigma_M(\zeta_{1,j}) = \zeta_{1,j} + c_{1,j}$, we obtain $\sigma_M(\zeta_{1,j})' = \zeta'_{1,j} = \sigma_M(\zeta'_{1,j})$. Now

$$\begin{aligned} \sigma_M(\zeta'_{i+1,j}) &= \sigma_M(f_j \zeta_{i,j+1}) \\ &= f_j \sigma_M(\zeta_{i,j+1}) \\ &= f_j \left(\zeta_{i,j+1} + \left(\sum_{t=1}^{i-1} c_{t,i+1+j-t} \zeta_{i-t,j+1} \right) + c_{i,j+1} \right) \\ &= \left(\zeta_{i+1,j} + \left(\sum_{t=1}^{i-1} c_{t,i+1+j-t} \zeta_{i+1-t,j} \right) + c_{i,j+1} \zeta_{1,j} \right)' \\ &= \sigma_M(\zeta_{i+1,j})'. \end{aligned}$$

Thus σ_M is a differential automorphism of the ring $F[g]$. Now extend σ_M to a differential field automorphism of the field of fractions E of $F[g]$. Thus G is isomorphic to the group $U(n+1, C)$. From propositions 2.3 and 2.4, we obtain that the differential operator

$$L(Y) := w(Y, 1, \zeta_{1,1}, \zeta_{2,1}, \dots, \zeta_{n,1}) / w(1, \zeta_{1,1}, \zeta_{2,1}, \dots, \zeta_{n,1})$$

has coefficients in the field F . Since V is the full set of solutions of the differential equation $L(Y) = 0$ and $F\langle V \rangle = E$, we conclude that E is a Picard-Vessiot extension of F for the differential operator $L(Y)$. It is also clear that one can realize any full unipotent subgroup of $GL(n, C)$ by suitably choosing a basis for V . \square

Proposition 3.2. *The differential ring $F[g]$ is the Picard-Vessiot ring of E .*

Proof. It suffices to show that $F[g]$ is a simple differential ring. Let I be a differential ideal of $F[g]$ and suppose that $I \cap F = \{0\}$. Let $S_{p,q} := \{\zeta_{i,j} | 1 \leq i \leq p-1\} \cup \{\zeta_{p,j} | 1 \leq j \leq q-1\}$. It is easy check that $F[S_{p,q}]$ is a differential ring. Choose p, q such that $I \cap F[S_{p,q}][\zeta_{p,q}] \neq \{0\}$ and that $I \cap F[S_{p,q}] = \{0\}$. Choose a non zero element $u \in I \cap F[S_{p,q}][\zeta_{p,q}]$ of smallest possible degree and write $u = \sum_{i=0}^n a_i \zeta_{p,q}^i$, where $a_n \neq 0$, $a_i \in F[S_{p,q}]$. Note that $n \geq 1$, degree of $a'_n u - a_n u' \leq n-1$ and that $a'_n u - a_n u' \in I$. Thus $a'_n u - a_n u' = 0$ and in particular, $a'_n a_{n-1} - a_n(n a_n f_q \zeta_{p-1,q+1} + a'_{n-1}) = 0$.

Then $(-a_{n-1}/na_n)' = f_q \zeta_{p-1,q+1}$ in E , which implies $\zeta_{p,q} + (a_{n-1}/na_n)$ is a new constant. This contradicts the fact that E is a no new constants extension of F . \square

4. EXAMPLES

4.1. Hyperlogarithms. Consider the differential field $F = \mathbb{C}(z)$ with the usual derivation d/dz . Let $\alpha_1, \dots, \alpha_n$ be distinct complex numbers. A hyperlogarithmic function is an iterated integral of the form

$$(4.1) \quad L(\alpha_1, \alpha_2, \dots, \alpha_n | z, z_0) := \int_{z_0}^z \int_{z_0}^{s_{n-1}} \cdots \int_{z_0}^{s_1} \frac{ds_0}{s_0 - \alpha_n} \cdots \frac{ds_{n-1}}{s_{n-1} - \alpha_1}$$

where z_0 is a fixed point and $z_0 \neq \alpha_n$, see [3]. Let $\mathcal{M}(U)$ be the field of meromorphic functions on U , where $z_0 \in U$ is a simply connected domain that does not contain α_i for any i . Let $f_j := 1/(z + \alpha_j)$. As noted in remark 1, these f_j 's satisfy the condition **C**. Using theorem 1.1 we may construct a Picard-vessiot extension $F(\zeta_{i,j} | 1 \leq i \leq n, 1 \leq j \leq n+1-i)$ for the differential operator

$$(4.2) \quad L(Y) = w(Y, 1, \zeta_{1,1}, \zeta_{2,1}, \dots, \zeta_{n,1}) / w(1, \zeta_{1,1}, \zeta_{2,1}, \dots, \zeta_{n,1}).$$

Define an F -algebra homomorphism $\phi := F[g] \rightarrow \mathcal{M}(U)$ such that $\phi(\zeta_{p,q}) = L(\alpha_q, \dots, \alpha_{p+q-1} | z, z_0)$. Then, one can see that this map commutes with the derivation d/dz and therefore $\ker \phi$ is a differential ideal of $F[g]$. Applying proposition 3.2, we conclude that $\ker \phi = \{0\}$ and thus ϕ is injective. This shows that the collection

$$\{L(\alpha_q, \alpha_{q+1}, \dots, \alpha_{p+q-1} | z, z_0) \mid 1 \leq i \leq p, 1 \leq j \leq n+1-p\}$$

of hyperlogarithms is algebraically independent over $\mathbb{C}(z)$.

We may compute the differential equation 4.2 using a Maple program, see [10]. Here, I will list differential equations for groups $U(n+1, \mathbb{C})$, when $n = 2, 3$ and 4.

$$n = 2 \quad \frac{d^3}{dz^3} + \frac{3z + \alpha_1 + 2\alpha_2}{(z + \alpha_1)(z + \alpha_2)} \frac{d^2}{dz^2} + \frac{d}{dz}.$$

$$\begin{aligned} n = 3 \quad & \frac{d^4}{dz^4} + \frac{6z^2 + (3\alpha_1 + 4\alpha_2 + 5\alpha_3)z + 2\alpha_3\alpha_1 + \alpha_2\alpha_1 + 3\alpha_3\alpha_2}{(z + \alpha_1)(z + \alpha_2)(z + \alpha_3)} \frac{d^3}{dz^3} \\ & + \frac{7z + \alpha_1 + 2\alpha_2 + 4\alpha_3}{(z + \alpha_1)(z + \alpha_2)(z + \alpha_3)} \frac{d^2}{dz^2} + \frac{1}{(z + \alpha_1)(z + \alpha_2)(z + \alpha_3)} \frac{d}{dz}. \end{aligned}$$

$$\begin{aligned}
 n = 4 \quad & \frac{d^5}{dz^5} + \frac{P_1 z^3 + P_2 z^2 + P_3 z + P_4}{S} \frac{d^4}{dz^4} + \frac{7z + \alpha_1 + 2\alpha_2 + 4\alpha_3}{S} \frac{d^3}{dz^3} \\
 & + \frac{P_5 z^2 + P_6 z + P_7}{S} \frac{d^2}{dz^2} + \frac{1}{S} \frac{d}{dz},
 \end{aligned}$$

where $P_1 = 10$, $P_2 = 9\alpha_4 + 8\alpha_3 + 7\alpha_2 + 6\alpha_1$, $P_3 = 5\alpha_1\alpha_4 + 4\alpha_1\alpha_3 + 5\alpha_2\alpha_5 + 6\alpha_2\alpha_4 + 3\alpha_2\alpha_1 + 7\alpha_3\alpha_4$, $P_4 = \alpha_1\alpha_2\alpha_3 + 2\alpha_1\alpha_2\alpha_4 + 3\alpha_1\alpha_3\alpha_4 + 4\alpha_2\alpha_3\alpha_4$, $P_5 = 25$, $P_6 = 7\alpha_1 + 10\alpha_2 + 14\alpha_3 + 19\alpha_4$, $P_7 = \alpha_1\alpha_2 + 2\alpha_1\alpha_3 + 3\alpha_2\alpha_3 + 9\alpha_3\alpha_4 + 6\alpha_2\alpha_4 + 4\alpha_1\alpha_4$ and $S = \prod_{i=1}^4 (z + \alpha_i)$.

REFERENCES

- [1] J. Ax, *On Schanuel's Conjectures*, Ann. of Math (2) **93** (1971), 252-268. MR **43**
- [2] A. Bialynicki-Birula, *On the inverse problem of Galois theory of differential fields*, Bull. Amer. Soc. **16** (1963), 960-964.
- [3] Matthieu Deneufchâtel, Gérard Henry Edmond Duchamp, Vincel Hoang Ngoc Minh, Allan I. Solomon, *Independence of hyperlogarithms over function fields via algebraic combinatorics*, arXiv:1101.4497v1 [math.CO].
- [4] A. Magid, **Lectures on Differential Galois Theory**, University Lecture Series. American Mathematical society 1994, 2nd edn.
- [5] M. van der Put, M. F. Singer, **Galois Theory of Linear Differential Equations**, **328**, Grundlehren der mathematischen Wissenschaften, Springer, Heidelberg, 2003.
- [6] C. Mitschi, M. Singer, *Connected Linear Groups as Differential Galois Groups*, 184 (1996), 333-361.
- [7] M. Rosenlicht, *On Liouville's Theory of Elementary Functions*, Pacific J. Math (2) **65** (1976), 485-492.
- [8] M. Rosenlicht, M. Singer, *On Elementary, Generalized Elementary, and Liouvillian Extension Fields*, Contributions to Algebra, (H. Bass et.al., ed.), Academic Press (1977) 329-342 .
- [9] V. Ravi Srinivasan, *Iterated Antiderivative Extensions*, Journal of Algebra (8) **324** (2020) 2042-2051.
- [10] V. Ravi Srinivasan, <http://andromeda.rutgers.edu/~ravisri/>.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, RUTGERS UNIVERSITY, NEWARK, NJ 07102.

E-mail address: ravisri@rutgers.edu